

A DISCONNECTED DEFORMATION SPACE OF RATIONAL MAPS

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ABSTRACT. Let $f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ be a postcritically finite rational map with postcritical set P . William Thurston showed that f induces a holomorphic pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ on the Teichmüller space $\mathcal{T}_P := \text{Teich}(\mathbb{P}^1, P)$. If f is not a flexible Lattès map, Thurston proved that σ_f has a unique fixed point. In his PhD thesis, Adam Epstein generalized Thurston's ideas and defined a deformation space associated to a rational map $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ where $A \subseteq B$, allowing for maps f which are not necessarily postcritically finite. By definition, the deformation space $\text{Def}_B^A(f) \subseteq \mathcal{T}_B$ is the locus where the pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ and the forgetful map $\sigma_A^B : \mathcal{T}_B \rightarrow \mathcal{T}_A$ agree. Using purely local arguments, Epstein showed that $\text{Def}_B^A(f)$ is a smooth analytic submanifold of \mathcal{T}_B of dimension $|B - A|$. In this article, we investigate the question of whether $\text{Def}_B^A(f)$ is connected. We exhibit a family of quadratic rational maps for which the associated deformation spaces are disconnected; in fact, each has infinitely many components.

1. INTRODUCTION

Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree at least 2 with critical set Ω . The postcritical set of f is $P := \bigcup_{n \geq 0} f^n(\Omega)$, the smallest forward invariant set that contains the critical values of f . We regard f as a self-map of the pointed space (\mathbb{P}^1, P) , and f is *postcritically finite* if $|P| < \infty$. In the 1980s, William Thurston established a topological characterization of rational maps which are postcritically finite [DH]. The proof of this theorem associates a dynamical system $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ to the dynamical system $f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$, where the space \mathcal{T}_P is the Teichmüller space of the pair (\mathbb{P}^1, P) , and the map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ is a holomorphic pullback map.

The map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ has a fixed point, and if $f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ is not of Lattès type, Thurston proved that some iterate of $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ is a strict contraction. The fixed point of $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ is therefore unique. This fact is known as *Thurston rigidity*.

Theorem 1.1 (W. Thurston, [DH]). *If $f : (\mathbb{P}^1, P) \rightarrow (\mathbb{P}^1, P)$ is a postcritically finite rational map which is not of Lattès type, then the associated pullback map $\sigma_f : \mathcal{T}_P \rightarrow \mathcal{T}_P$ has a unique fixed point.*

In 1993, Adam Epstein generalized Thurston's ideas to rational maps $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ which are not necessarily postcritically finite [E]. In this setting, he imposed the following conditions on the sets A and B :

- (1) A and B are finite, each containing at least 3 points,
- (2) B contains the critical values of f ,
- (3) $f(A) \subseteq B$, and
- (4) $A \subseteq B$.

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The first three conditions are necessary to define a pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$, and the last condition is necessary to define a forgetful map $\sigma_A^B : \mathcal{T}_B \rightarrow \mathcal{T}_A$ (see Section 2 for definitions).

1.1. The deformation space. Let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be a rational map for which the sets A and B satisfy the conditions (1)-(4) above. Epstein defined the associated *deformation space*

$$\text{Def}_B^A(f) := \{\tau \in \mathcal{T}_B \mid \sigma_f(\tau) = \sigma_A^B(\tau)\}.$$

In other words, $\text{Def}_B^A(f)$ is the *equalizer* of the two maps $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ and $\sigma_A^B : \mathcal{T}_B \rightarrow \mathcal{T}_A$.

Theorem 1.2 (A. Epstein, [E]). *Let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be a rational map that is not of Lattès type. The deformation space $\text{Def}_B^A(f)$ is a smooth analytic submanifold of \mathcal{T}_B of dimension $|B - A|$.*

If $A = B$ in the theorem above, then $\text{Def}_B^A(f)$ is equal to the set of fixed points of σ_f , so by Theorem 1.1, $\text{Def}_B^A(f)$ consists of a single point [BCT].

The proof of Theorem 1.2 is purely local and reveals nothing about the global structure of $\text{Def}_B^A(f)$. In this article, we prove that $\text{Def}_B^A(f)$ is disconnected for the following quadratic rational maps.

1.2. A family of quadratic rational maps. Let M_2 be the moduli space of quadratic rational maps: the space of quadratic rational maps up to conjugation by Möbius transformations. The locus $\text{Per}_4(0) \subseteq M_2$ consists of conjugacy classes of maps with a superattracting cycle of period 4. That is, for $\langle f \rangle \in \text{Per}_4(0)$ the map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ has a critical point which is periodic of period 4. Define $\text{Per}_4(0)^* \subseteq \text{Per}_4(0)$ to be the subset consisting of $\langle f \rangle$ for which the superattracting 4-cycle contains only one critical point of f .

Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ represent an element of $\text{Per}_4(0)^*$. Define the set A to be the set of points in the superattracting 4-cycle, and define the set $B := A \cup \text{cv}(f)$, where $\text{cv}(f)$ is the set of critical values of f . Consider the associated deformation space $\text{Def}_B^A(f)$. By Theorem 1.2, $\text{Def}_B^A(f)$ is a 1-dimensional submanifold of \mathcal{T}_B which is 2-dimensional. The following theorem is our main result.

Theorem 1.3 (Main Theorem). *For $\langle f \rangle \in \text{Per}_4(0)^*$, the space $\text{Def}_B^A(f)$ has infinitely many connected components.*

The proof of Theorem 1.3 reduces to a comparison of the stabilizer of $\text{Def}_B^A(f)$ in the automorphism group of Teichmüller space with the stabilizer of a connected component of $\text{Def}_B^A(f)$.

The Teichmüller space \mathcal{T}_B is the universal cover of the moduli space \mathcal{M}_B (the moduli space of ordered points on the Riemann sphere), and the group of deck transformations of the universal covering map $\mathcal{T}_B \rightarrow \mathcal{M}_B$ is naturally isomorphic to Mod_B , the pure mapping class group of (\mathbb{P}^1, B) . We introduce a certain subgroup $S_f \subseteq \text{Mod}_B$ called the special liftables. This group, which is the trivial group in the dynamical setting where $A = B$, plays a fundamental role in our study of $\text{Def}_B^A(f)$. In Proposition 2.1, we show that S_f consists of elements which restrict to automorphisms of $\text{Def}_B^A(f)$. The space \mathcal{T}_B comes with a canonical basepoint $\otimes \in \text{Def}_B^A(f)$. Let $\overline{\otimes}$ be the image of \otimes under the quotient map $\text{Def}_B^A(f) \rightarrow \text{Def}_B^A(f)/S_f$, and consider the homomorphism

$$\bar{\iota}_* : \pi_1(\text{Def}_B^A(f)/S_f, \overline{\otimes}) \longrightarrow \pi_1(\mathcal{T}_B/S_f, \overline{\otimes})$$

induced by inclusion. We establish Theorem 1.3 by showing that the index of the image of \bar{l}_* in $\pi_1(\mathcal{T}_B/S_f, \overline{\otimes})$ is infinite.

Remark 1.1. The authors learned that T. Firsova, J. Kahn, and N. Selinger proved a related result in [FKS]. Their work was completed independently, and at the same time that the authors proved Theorem 1.3.

Outline. The paper is organized as follows. In Section 2, we recall basic definitions and introduce subgroups $S_f \subseteq L_f \subseteq \text{Mod}_B$ associated to $\text{Def}_B^A(f)$. In Section 3, we look at the image of $\text{Def}_B^A(f)$ in an intermediate covering space \mathcal{W} of \mathcal{M}_B , and reduce the problem of connectivity to a question about covering spaces and group actions. Theorem 1.3 is proved in Section 4. Relevant general techniques for computing the fundamental group of complements of plane curves, in particular real line arrangements, are recalled in Appendix A.

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2. PRELIMINARIES

In this section, we review definitions and translate the problem of connectivity of $\text{Def}_B^A(f)$ to a problem about groups.

Recall that a Riemann surface is a connected oriented topological surface together with a *complex structure*: a maximal atlas of charts $\phi : U \rightarrow \mathbb{C}$ with holomorphic overlap maps. For a given oriented, compact topological surface X , we denote the set of all complex structures on X by $\mathcal{C}(X)$. An orientation-preserving branched covering map $f : X \rightarrow Y$ induces a map $f^* : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$; in particular, for any orientation-preserving homeomorphism $\psi : X \rightarrow X$, there is an induced map $\psi^* : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$.

Let $A \subseteq X$ be finite. The Teichmüller space of (X, A) is

$$\text{Teich}(X, A) := \mathcal{C}(X) / \sim_A$$

where $c_1 \sim_A c_2$ if and only if $c_1 = \psi^*(c_2)$ for some orientation-preserving homeomorphism $\psi : X \rightarrow X$ which is isotopic to the identity relative to A .

2.1. The forgetful map. If $A \subseteq B \subseteq X$ are finite sets then for $c_1, c_2 \in \mathcal{C}(X)$, if $c_1 \sim_B c_2$ then $c_1 \sim_A c_2$. This allows us to define the *forgetful map* $\sigma_A^B : \text{Teich}(X, B) \rightarrow \text{Teich}(X, A)$ taking the equivalence class of c in $\text{Teich}(X, B)$ to the equivalence class of c in $\text{Teich}(X, A)$.

2.2. The pullback map. In view of the homotopy-lifting property, if

- $B \subseteq Y$ is finite and contains the critical values of f , and
- $f(A) \subseteq B$,

then $f^* : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$ descends to a well-defined map σ_f between the corresponding Teichmüller spaces.

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{f^*} & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \text{Teich}(Y, B) & \xrightarrow{\sigma_f} & \text{Teich}(X, A) \end{array}$$

This map is known as the *pullback map* induced by f .

2.3. Genus zero. Let X be the 2-sphere and fix an identification $X = \mathbb{P}^1$. Then the Teichmüller space $\mathcal{T}_A := \text{Teich}(X, A)$ comes with a *canonical basepoint* $\otimes := [\text{id}]$, and we use the Uniformization Theorem to obtain the following description of \mathcal{T}_A . Given a finite set $A \subseteq \mathbb{P}^1$ containing at least 3 points, the space \mathcal{T}_A is the quotient of the space of all orientation-preserving homeomorphisms $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by the equivalence relation \sim where $\phi_1 \sim \phi_2$ if there exists a Möbius transformation ν such that $\nu \circ \phi_1 = \phi_2$ on A , and $\nu \circ \phi_1$ is isotopic to ϕ_2 relative to A . The Teichmüller space \mathcal{T}_A is a complex manifold of dimension $|A| - 3$.

The *moduli space* of the pair (\mathbb{P}^1, A) is the space of injective maps $\varphi : A \hookrightarrow \mathbb{P}^1$ modulo postcomposition by Möbius transformations. The moduli space is a complex manifold isomorphic to the complement of finitely many hyperplanes in $\mathbb{C}^{|A|-3}$. We denote the moduli space as \mathcal{M}_A .

If ϕ represents an element of the Teichmüller space \mathcal{T}_A , the restriction $\phi \mapsto \phi|_A$ induces a universal covering map $\mathcal{T}_A \rightarrow \mathcal{M}_A$ which is a local biholomorphism with respect to the complex structures on \mathcal{T}_A and \mathcal{M}_A . The group of deck transformations is naturally isomorphic to the *pure mapping class group* Mod_A , the quotient of the group of orientation-preserving homeomorphisms $h : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ fixing A pointwise by the subgroup of such maps isotopic to the identity relative to A . This group acts freely and properly discontinuously on \mathcal{T}_A .

The forgetful map $\sigma_A^B : \mathcal{T}_B \rightarrow \mathcal{T}_A$ is a holomorphic surjective submersion and descends to the corresponding forgetful map on moduli space $\mu_A^B : \mathcal{M}_B \rightarrow \mathcal{M}_A$.

2.4. Admissible rational maps. We will say that the rational map $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ is *admissible* if:

- A and B are finite sets, each containing at least 3 points,
- B contains the critical values of f , and
- $f(A) \subseteq B$.

Let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be an admissible rational map, let $\tau \in \mathcal{T}_B$ and let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a homeomorphism representing τ . By the Uniformization Theorem, there exist

- a homeomorphism $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ representing $\tau' := \sigma_f(\tau)$, and
- a rational map $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$,

such that the following diagram commutes.

$$\begin{array}{ccc} (\mathbb{P}^1, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ \downarrow f & & \downarrow F \\ (\mathbb{P}^1, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

Conversely, if we have such a commutative diagram with F holomorphic, then

$$\sigma_f(\tau) = \tau'$$

where $\tau \in \mathcal{T}_B$ and $\tau' \in \mathcal{T}_A$ are the equivalence classes of ϕ and ψ respectively. The map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ is holomorphic.

2.5. The liftables. Let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be an admissible rational map, and let $h : (\mathbb{P}^1, B) \rightarrow (\mathbb{P}^1, B)$ be an orientation-preserving homeomorphism that fixes B pointwise. We say that h is *liftable* if there is an orientation-preserving homeomorphism $h' : (\mathbb{P}^1, A) \rightarrow$

(\mathbb{P}^1, A) fixing A pointwise so that $h \circ f = f \circ h'$. The lift h' is only determined up to deck transformations of f that preserve A . Let

$$L_f := \{[h] \in \text{Mod}_B \mid h \text{ is liftable}\}$$

be the subgroup of *liftable mapping classes*, or just the *liftables* associated to f . As proved in Proposition 3.1 of [KPS], the subgroup L_f has finite index in Mod_B , and there is a *lifting homomorphism*

$$\begin{aligned} \Phi_f : L_f &\rightarrow \text{Mod}_A \\ [h] &\mapsto [h'] \end{aligned}$$

which is well-defined because the action of Mod_A is free on \mathcal{T}_A .

2.6. The special liftables. Let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be an admissible rational map. We now work in the case where domain and range of f are identified and $A \subseteq B$. We define a subgroup $S_f \subseteq L_f$ that preserves $\text{Def}_B^A(f)$.

Because $A \subseteq B$, there is a *forgetful homomorphism* $\Phi_A^B : \text{Mod}_B \rightarrow \text{Mod}_A$ corresponding to forgetting points in $B - A$. The subgroup of *special liftable mapping classes*, or just the *special liftables* associated to $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ is defined to be

$$S_f := \{g \in L_f \mid \Phi_f(g) = \Phi_A^B(g)\},$$

the equalizer of the homomorphisms Φ_f and Φ_A^B . Not much is known about this subgroup in general. In the purely dynamical setting where $A = B$ and f is not a Lattès map, it follows from Theorem 1.1 that S_f is the trivial group; in particular, it has infinite index in Mod_B .

Proposition 2.1. *Let $g \in L_f$. The following are equivalent:*

- (1) $g \cdot \text{Def}_B^A(f) \cap \text{Def}_B^A(f) \neq \emptyset$,
- (2) $g \cdot \text{Def}_B^A(f) = \text{Def}_B^A(f)$, and
- (3) $g \in S_f$.

Proof. The proof is purely formal and uses the following two equations:

- for all $h \in \text{Mod}_B$ and for all $\tau \in \mathcal{T}_B$,

$$\sigma_A^B(h \cdot \tau) = \Phi_A^B(h) \cdot \sigma_A^B(\tau),$$

- for all $h \in L_f$ and for all $\tau \in \mathcal{T}_B$,

$$\sigma_f(h \cdot \tau) = \Phi_f(h) \cdot \sigma_f(\tau).$$

Let $\tau \in \text{Def}_B^A(f)$. Since $g \in L_f$, we have

$$\sigma_f(g \cdot \tau) = \sigma_A^B(g \cdot \tau) \quad \text{if and only if} \quad \Phi_f(g) \cdot \sigma_f(\tau) = \Phi_A^B(g) \cdot \sigma_A^B(\tau).$$

Because $\tau \in \text{Def}_B^A(f)$, $\sigma_f(\tau) = \sigma_A^B(\tau)$. Since elements of Mod_B have no fixed points, we have, for any $\tau' \in \mathcal{T}_A$,

$$\Phi_f(g) \cdot \tau' = \Phi_A^B(g) \cdot \tau' \quad \text{if and only if} \quad \Phi_f(g) = \Phi_A^B(g) \quad \text{if and only if} \quad g \in S_f.$$

□

3. QUOTIENTS

Throughout the rest of this paper, let $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ be an admissible rational map so that $A \subseteq B$, and assume that f is not of Lattès type.

The quotient $\mathcal{W} := \mathcal{T}_B/\mathcal{L}_f$ is a connected complex manifold of dimension $|B| - 3$, and the quotient $\text{Def}_B^A(f)/S_f$ is a (possibly disconnected) complex submanifold of \mathcal{T}_B/S_f of dimension $|B - A|$. The space \mathcal{W} comes equipped with maps

$$\mu_B : \mathcal{W} \rightarrow \mathcal{M}_B \quad \text{and} \quad \mu_A : \mathcal{W} \rightarrow \mathcal{M}_A$$

so that the diagram below, excluding the dashed arrow, commutes.

$$(1) \quad \begin{array}{ccccc} \text{Def}_B^A(f) & \xhookrightarrow{\iota} & \mathcal{T}_B & \xrightarrow{\sigma_f} & \mathcal{T}_A \\ \downarrow & & \downarrow q & & \downarrow \\ \text{Def}_B^A(f)/S_f & \xhookrightarrow{\bar{\iota}} & \mathcal{T}_B/S_f & & \\ & & \downarrow \omega & & \\ & & \mathcal{W} := \mathcal{T}_B/\mathcal{L}_f & \xrightarrow{\mu_A} & \\ & & \downarrow \mu_B & \searrow & \\ & & \mathcal{M}_B & \dashrightarrow^{\mu_A^B} & \mathcal{M}_A \end{array}$$

The map $\mu_B : \mathcal{W} \rightarrow \mathcal{M}_B$ is a finite cover and the map $\mu_A : \mathcal{W} \rightarrow \mathcal{M}_A$ can be just about anything; for example, it may be constant [BEKP]. It follows from Proposition 2.1, that $\omega \circ \bar{\iota}$ is injective. Let $\mathcal{V} \subseteq \mathcal{W}$ denote its image. The space \mathcal{V} is a subset of the equalizer of the pair of maps $\mu_A : \mathcal{W} \rightarrow \mathcal{M}_A$ and $\mu_A^B \circ \mu_B : \mathcal{W} \rightarrow \mathcal{M}_A$; that is

$$(2) \quad \mathcal{V} \subseteq \{w \in \mathcal{W} \mid \mu_A(w) = \mu_A^B \circ \mu_B(w)\}.$$

3.1. Fundamental groups. In this section, we review some notions from the theory of covering spaces that will be required for the proof of Theorem 1.3.

The canonical basepoint \otimes of \mathcal{T}_B lies in $\text{Def}_B^A(f)$ and determines basepoints

$$\otimes_{\mathcal{V}} \in \mathcal{V} \subseteq \mathcal{W}, \quad \otimes_B \in \mathcal{M}_B, \quad \text{and} \quad \otimes_A \in \mathcal{M}_A.$$

Let $i : \mathcal{V} \hookrightarrow \mathcal{W}$ be the inclusion. We have the following commutative diagram.

$$\begin{array}{ccc} (\text{Def}_B^A(f)/S_f, \otimes) & \xhookrightarrow{\bar{\iota}} & (\mathcal{T}_B/S_f, \otimes) \\ \downarrow \omega \circ \bar{\iota} & & \downarrow \omega \\ (\mathcal{V}, \otimes_{\mathcal{V}}) & \xhookrightarrow{i} & (\mathcal{W}, \otimes_{\mathcal{V}}) \end{array}$$

It follows from Proposition 2.1 the map $\omega \circ \bar{\iota}$ is a homeomorphism.

The induced diagram on fundamental groups is:

$$\begin{array}{ccc} \pi_1(\text{Def}_B^A(f)/S_f, \overline{\otimes}) & \xrightarrow{\bar{\iota}_*} & \pi_1(\mathcal{T}_B/S_f, \overline{\otimes}) \\ \downarrow (\omega \circ \bar{\iota})_* & & \downarrow \omega_* \\ \pi_1(\mathcal{V}, \otimes_{\mathcal{V}}) & \xrightarrow{i_*} & \pi_1(\mathcal{W}, \otimes_{\mathcal{V}}) \end{array}$$

where $(\omega \circ \bar{\iota})_*$ is an isomorphism. If \mathcal{V} is disconnected, $\pi_1(\mathcal{V}, \otimes_{\mathcal{V}})$ is defined to be the fundamental group of the connected component containing $\otimes_{\mathcal{V}}$.

As a consequence we have the following result.

Proposition 3.1. *The map $\bar{\iota}_*$ is injective (respectively surjective) if and only if i_* is injective (respectively surjective).*

Proposition 3.2. *The basepoints determined by \otimes define identifications*

$$\begin{aligned} \text{Mod}_B &= \pi_1(\mathcal{M}_B, \otimes_B) \\ \text{Mod}_A &= \pi_1(\mathcal{M}_A, \otimes_A) \\ L_f &= \pi_1(\mathcal{W}, \otimes_{\mathcal{V}}) \\ S_f &= \omega_*(\pi_1(\mathcal{T}_B/S_f, \overline{\otimes})) \subseteq L_f \end{aligned}$$

such that

$$\Phi_f = (\mu_A)_*, \quad \Phi_A^B = (\mu_A^B)_*, \quad \text{and} \quad \Phi_A^B|_{L_f} = (\mu_A^B \circ \mu_B)_*.$$

Proof. The proofs that

$$\Phi_A^B = (\mu_A^B)_* \quad \text{and} \quad \Phi_A^B|_{L_f} = (\mu_A^B \circ \mu_B)_*$$

are immediate from the definitions and from the identifications above. We will prove that $\Phi_f = (\mu_A)_*$. Let γ be an oriented loop in \mathcal{W} based at $\otimes_{\mathcal{V}}$, and set $\alpha := \mu_A(\gamma)$, an oriented loop in \mathcal{M}_A based at \otimes_A . By the identifications above, $[\gamma] \in \pi_1(\mathcal{W}, \otimes_{\mathcal{V}})$ determines some $g \in L_f$, and $[\alpha] \in \pi_1(\mathcal{M}_A, \otimes_A)$ determines some $h \in \text{Mod}_A$. Because Diagram (1) commutes, we have

$$\sigma_f(g \cdot \otimes) = h \cdot \sigma_f(\otimes).$$

Because $g \in L_f$, we must have $h = \Phi_f(g)$. □

Proposition 3.3. *We have*

$$S_f = \{\gamma \in L_f \mid (\mu_A)_*(\gamma) = (\mu_A^B \circ \mu_B)_*(\gamma)\}.$$

Proof. This is a direct consequence of Proposition 3.2. □

3.2. A criterion for connectivity. In our particular family of examples, \mathcal{V} is connected (see Section 4.2), and hence the question arises whether or not $\text{Def}_B^A(f)$ is also connected. Let

$$E_f := i_*(\pi_1(\mathcal{V}, \otimes_{\mathcal{V}})) \subseteq S_f.$$

In general we have the following.

Proposition 3.4. *Suppose that \mathcal{V} is connected. There is a bijection between the connected components of $\text{Def}_B^A(f)$ and the (left) cosets of E_f in S_f .*

Proof. The fibers of the covering map $p := \omega \circ q \circ \iota : \text{Def}_B^A(f) \rightarrow \mathcal{V}$ are in bijective correspondence with S_f . In particular, path-lifting at \otimes defines a bijection

$$\beta_S : S_f \rightarrow p^{-1}(\otimes_{\mathcal{V}}),$$

and an injection

$$\beta_E : E_f \rightarrow p^{-1}(\otimes_{\mathcal{V}}),$$

whose image is contained in a connected component of $p^{-1}(\mathcal{V})$.

For two elements of $\gamma, \gamma' \in S_f$, $\beta(\gamma)$ and $\beta(\gamma')$ lie in the same component of $\text{Def}_B^A(f)$ if and only if $\gamma' = \gamma \circ \alpha$ for some $\alpha \in E_f$, that is, if and only if the cosets are equal:

$$\gamma' \cdot E_f = \gamma \cdot E_f.$$

□

As an easy consequence, we have the following.

Corollary 3.5. *Suppose that \mathcal{V} is connected. Then $\text{Def}_B^A(f)$ is connected if and only if $E_f = S_f$.*

It is also of interest to consider the topology of the components of $\text{Def}_B^A(f)$.

Proposition 3.6. *The components of $\text{Def}_B^A(f)$ are simply-connected if and only if i_* is injective.*

Proof. By the theory of covering spaces, each connected component of $\text{Def}_B^A(f)$, is a covering space of \mathcal{V} with fundamental group equal to the kernel of i_* . □

3.3. Equalizers and fundamental groups. We pause here to reformulate the problem we are solving for our family of maps $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$. In Proposition 4.2, we will prove that

$$\mathcal{V} = \{w \in \mathcal{W} \mid \mu_A^B \circ \mu_B(w) = \mu_A(w)\}$$

that is, \mathcal{V} is the equalizer of the maps $\mu_A^B \circ \mu_B : \mathcal{W} \rightarrow \mathcal{M}_A$ and $\mu_A : \mathcal{W} \rightarrow \mathcal{M}_A$. By Proposition 3.3, we have

$$S_f = \{\gamma \in \pi_1(\mathcal{W}, \otimes_{\mathcal{V}}) \mid (\mu_A^B \circ \mu_B)_*(\gamma) = (\mu_A)_*(\gamma)\}.$$

The space $\text{Def}_A^B(f)$ is connected if and only if $E_f = S_f$. That is, if the fundamental group of the equalizer equals the equalizer of the fundamental groups. For our particular examples $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$, we will prove that that E_f has infinite index in S_f . By Proposition 3.4, this will establish Theorem 1.3.

4. THE PROOF OF THEOREM 1.3

Let $\langle f \rangle \in \text{Per}_4(0)^*$. By conjugating with a Möbius transformation, we may suppose that f has a superattracting cycle of the form

$$(3) \quad \begin{array}{ccccccc} 0 & \xrightarrow{2} & \infty & \longrightarrow & 1 & \longrightarrow & a \\ & & \searrow & & \nearrow & & \\ & & & & & & \end{array}$$

As evident by the work that follows, our results hold for any map f representing an element of $\text{Per}_4(0)^*$ with appropriately defined sets A and B . For the sake of presentation however,

we will work with a concrete example, so that in our coordinates the basepoint $\otimes_{\mathcal{V}} \in \mathbb{R}$. We will work with

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \quad \text{given by} \quad f : z \mapsto \frac{(4z-3)(z+2)}{4z^2}.$$

The map f has a superattracting cycle of the form in Line (3) for $a = 3/4$. The critical points of f are $\{0, 12/5\}$, and the critical values of f are $\{\infty, 121/96\}$. Define the set $A = \{0, 1, \infty, 3/4\}$, and the set $B = A \cup \{121/96\}$. By Theorem 1.2, $\text{Def}_B^A(f)$ is a 1-dimensional submanifold of a 2-dimensional Teichmüller space \mathcal{T}_B . The space \mathcal{T}_A is 1-dimensional. We compute the spaces \mathcal{W} and \mathcal{V} for this particular f .

We set our coordinates before doing computations. Given $[\alpha] \in \mathcal{M}_A$ and $[\beta] \in \mathcal{M}_B$, let ν_α and ν_β be Möbius transformations such that $\nu_\alpha \circ \alpha$ and $\nu_\beta \circ \beta$ are the identity on $\{0, 1, \infty\}$. Then we define coordinates:

$$\begin{aligned} \mathcal{M}_A &\rightarrow \mathbb{C} - \{0, 1\} \\ [\alpha] &\mapsto x := (\nu_\alpha \circ \alpha)(3/4), \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_B &\rightarrow \mathbb{C}^2 - \{y=0, y=1, y=z, z=0, z=1\} \\ [\beta] &\mapsto (y, z) := ((\nu_\beta \circ \beta)(3/4), (\nu_\beta \circ \beta)(121/96)). \end{aligned}$$

4.1. The space \mathcal{W} . By Lemma 2.5 in [K], a point in \mathcal{W} consists of (x, y, z, F) where F is a rational map

$$F : (\mathbb{P}^1, \{0, 1, \infty, x\}) \rightarrow (\mathbb{P}^1, \{0, 1, \infty, y, z\})$$

satisfying the combinatorial conditions below, where the marked points are distinct:

$$\begin{array}{ccccc} 0 & \infty & 1 & x & * \\ \downarrow 2 & \downarrow & \downarrow & \downarrow & \downarrow 2 \\ \infty & 1 & y & 0 & z \end{array}$$

that is, 0 is a critical point of F , $\text{cv}(F) = \{\infty, z\}$, and

$$F(0) = \infty, \quad F(\infty) = 1, \quad F(1) = y, \quad \text{and} \quad F(x) = 0.$$

As can easily be verified, such a rational map $F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ must be of the following form:

$$F(t) = \frac{(x-t)(-tx+y+t+x-1)}{(x-1)t^2}, \quad \text{where} \quad z = \frac{(-x^2+y+2x-1)^2}{4x(y-1+x)(1-x)}.$$

Note that the map F has a superattracting cycle of the form in Line (3) if and only if $x = y$.

There is an isomorphism

$$\mathcal{W} \rightarrow \mathbb{C}^2 - \Delta \quad \text{given by} \quad (x, y, z, F) \mapsto (x, y)$$

where Δ consists of all “forbidden” pairs (x, y) leading to collisions of points in $\{0, 1, \infty, z\}$, or collisions of points in $\{0, 1, \infty, y, z\}$. The set Δ can be computed explicitly:

$$\begin{aligned} \Delta = \{ (x, y) \in \mathbb{C}^2 \mid &x = 0, y = 0, y = 1, x = 1, y - 1 + x = 0, x^2 - y - 2x + 1 = 0, \\ &x^2 + y - 1 = 0, \text{ or } 2xy + x^2 - y - 2x + 1 = 0 \}. \end{aligned}$$

We will use (x, y) as coordinates on \mathcal{W} . Let $\text{proj}_x : \mathbb{C}^2 \rightarrow \mathbb{C}$ (respectively, $\text{proj}_y : \mathbb{C}^2 \rightarrow \mathbb{C}$) be projection of \mathbb{C}^2 onto the x -coordinate (respectively, y -coordinate).

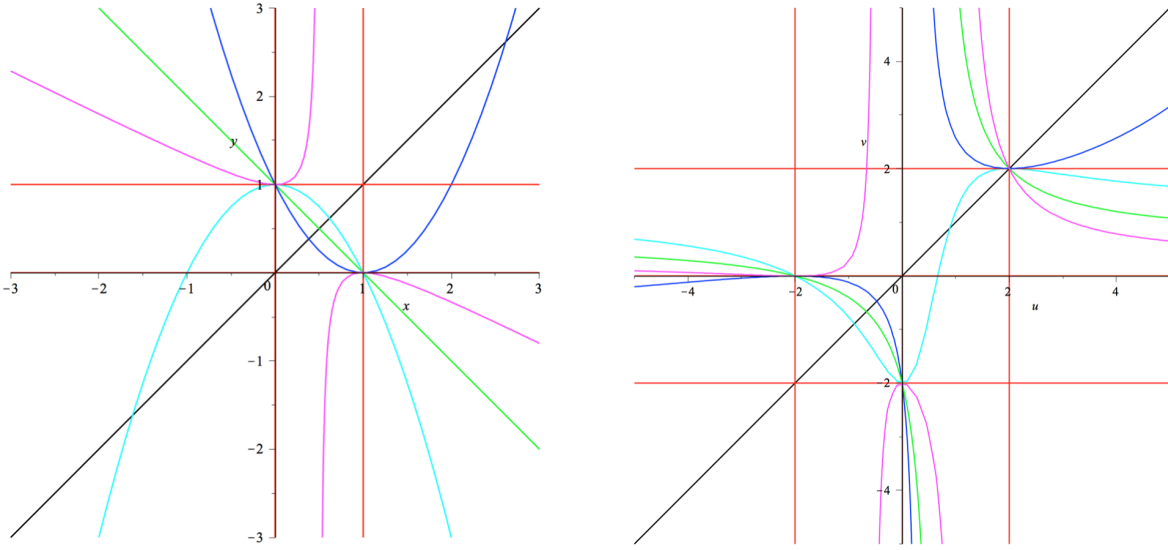


FIGURE 1. On the left is the space \mathcal{W} drawn in \mathbb{R}^2 in (x, y) -coordinates; it is the complement of the curves in Δ , which are drawn in color. The black diagonal line is \mathcal{V} , and it intersects Δ in 10 points (two of which are complex conjugates and one which is at (∞, ∞)). On the right is a picture of \mathcal{W} near $(x, y) = (\infty, \infty)$ drawn in (u, v) -coordinates, where $x = \frac{2u}{u-2}$ and $y = \frac{2v}{v-2}$.

Proposition 4.1. *The maps μ_A , μ_A^B and μ_B can be expressed in these coordinates as follows: the map μ_A is given by*

$$\begin{aligned} \mu_A : \mathcal{W} &\rightarrow \mathcal{M}_A \\ (x, y) &\mapsto x; \end{aligned}$$

the map μ_B is a degree 4 covering map,

$$\begin{aligned} \mu_B : \mathcal{W} &\rightarrow \mathcal{M}_B \\ (x, y) &\mapsto (y, z) \end{aligned}$$

where

$$z = \frac{(-x^2 + y + 2x - 1)^2}{4x(y - 1 + x)(1 - x)};$$

and the map μ_A^B is given by

$$\begin{aligned} \mu_A^B : \mathcal{M}_B &\rightarrow \mathcal{M}_A \\ (y, z) &\mapsto y. \end{aligned}$$

Thus we have

$$\mu_A = \text{proj}_x|_{\mathcal{W}} \quad \mu_A^B \circ \mu_B = \text{proj}_y|_{\mathcal{W}}.$$

And in these coordinates, $\otimes_{\mathcal{V}} = (3/4, 3/4)$.

Proof. All of these assertions follow directly from the definitions. □

4.2. **The space \mathcal{V} .** We establish some properties of \mathcal{V} in our particular setting.

Proposition 4.2. *In these coordinates, the space $\mathcal{V} \subseteq \mathcal{W}$ is equal to the diagonal; that is,*

$$\mathcal{V} = \{(x, y) \in \mathcal{W} \mid x = y\},$$

and \mathcal{V} is isomorphic to $\text{Per}_4(0)^$.*

Proof. For purposes of this proof, set $D := \{(x, y) \in \mathcal{W} \mid x = y\}$. By Line (2), $\mathcal{V} \subseteq D$. Let $\gamma : [0, 1] \rightarrow D$ be a path with the property that $\gamma(0) = \otimes_{\mathcal{V}}$. Because $\omega \circ q : \mathcal{T}_B \rightarrow \mathcal{W}$ is a covering map, there is a unique lift $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{T}_B$ with $\tilde{\gamma}(0) = \otimes$. We prove that $\tilde{\gamma}(t) \in \text{Def}_B^A(f)$ for all $t \in [0, 1]$, establishing the result. Let $\phi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a homeomorphism representing $\tilde{\gamma}(t)$, which satisfies

$$\phi_t|_{\{0,1,\infty\}} = \text{id}|_{\{0,1,\infty\}}.$$

There is a homeomorphism $\psi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ representing $\sigma_f(\tilde{\gamma}(t))$, which satisfies

$$\psi_t|_{\{0,1,\infty\}} = \text{id}|_{\{0,1,\infty\}},$$

and a rational map $F_t : (\mathbb{P}^1, \psi_t(A)) \rightarrow (\mathbb{P}^1, \phi_t(B))$ such that the diagram

$$\begin{array}{ccc} (\mathbb{P}^1, A) & \xrightarrow{\psi_t} & (\mathbb{P}^1, \psi_t(B)) \\ \downarrow f & & \downarrow F_t \\ (\mathbb{P}^1, B) & \xrightarrow{\phi_t} & (\mathbb{P}^1, \phi_t(B)) \end{array}$$

commutes. The path $\tilde{\gamma}$ defines an isotopy from $\phi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ to the identity, and the path $\sigma_f \circ \tilde{\gamma}$ defines an isotopy from $\psi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ to the identity. The composition $\phi_t^{-1} \circ \psi_t : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is isotopic to the identity relative to A because $\gamma(t) \in D$ and therefore $\psi_t|_A = \phi_t|_A$. This implies $[\psi_t] = \sigma_A^B([\phi_t])$; that is, $\sigma_f^B(\tilde{\gamma}(t)) = \sigma_f(\tilde{\gamma}(t))$.

It follows by construction that $D = \text{Per}_4(0)^*$. □

Corollary 4.3. *The space \mathcal{V} is connected, and the restrictions*

$$\mu_B|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{M}_B \quad \text{and} \quad \mu_A|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{M}_A$$

are injective.

Proof. By the above choice of coordinates, \mathcal{V} is isomorphic to \mathbb{P}^1 with 10 punctures (see Figure 1). It is easily verified that the restrictions $\mu_B|_{\mathcal{V}}$ and $\mu_A|_{\mathcal{V}}$ are injective. □

As a consequence, the group S_f is equal to the stabilizer of $\text{Def}_B^A(f)$ for our family of examples $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ as proven in Proposition 4.4.

Proposition 4.4. *The subgroup of Mod_B consisting of elements which restrict to automorphisms of $\text{Def}_B^A(f)$ is S_f .*

Proof. Let $G_f \subseteq \text{Mod}_B$ be the subgroup of elements which restrict to automorphisms of $\text{Def}_B^A(f)$. By Proposition 2.1, $S_f \subseteq G_f$. Consider the commutative diagram:

$$\begin{array}{ccccc}
\text{Def}_B^A(f) & \xhookrightarrow{\iota} & \mathcal{T}_B & & \\
\downarrow & & \downarrow q & & \\
\text{Def}_B^A(f)/S_f & \xhookrightarrow{\bar{\iota}} & \mathcal{T}_B/S_f & & \\
\downarrow & & \downarrow & \searrow \omega & \\
\text{Def}_B^A(f)/G_f & \xhookrightarrow{\quad} & \mathcal{T}_B/G_f & & \mathcal{T}_B/L_f \\
\downarrow & & \downarrow & \swarrow \mu_B & \\
& & \mathcal{M}_B & &
\end{array}$$

The map

$$\mu_B \circ \omega \circ \bar{\iota} : \text{Def}_B^A(f)/S_f \rightarrow \mathcal{M}_B$$

is injective and hence the map

$$\text{Def}_B^A(f)/S_f \rightarrow \text{Def}_B^A(f)/G_f$$

is a homeomorphism. Thus $S_f = G_f$. \square

At this point, we would like to compare E_f , and S_f , but the fundamental groups of the spaces involved are rather complicated. Instead, we will include the space \mathcal{W} into a somewhat simpler space $\widehat{\mathcal{W}}$ where the fundamental group is easier to understand, $j : \mathcal{W} \hookrightarrow \widehat{\mathcal{W}}$. We will compare the corresponding groups $\widehat{E} := j_*(E_f)$ and $\widehat{S} := j_*(S_f)$ in $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$.

4.3. A simpler space. Define

$$\widehat{\mathcal{W}} := \mathbb{C}^2 - \{x = 0, y = 0, x = 1, y = 1, x + y = 1\},$$

the complement in \mathbb{C}^2 of the lines C, R_1, R_2, S_1, S_2 in Figure 2, and define $\widehat{\mathcal{V}} := \{(x, y) \in \widehat{\mathcal{W}} \mid y = x\}$; it is isomorphic to $\mathbb{P}^1 - \{0, 1/2, 1, \infty\}$. Let $j : \mathcal{W} \hookrightarrow \widehat{\mathcal{W}}$ and $\widehat{i} : \widehat{\mathcal{V}} \hookrightarrow \widehat{\mathcal{W}}$ be the inclusion maps. By functoriality of fundamental groups, we have

$$\widehat{E} = \widehat{i}_*(\pi_1(\widehat{\mathcal{V}}, \otimes_{\mathcal{V}})).$$

Proposition 4.5. *The map*

$$j_* : \pi_1(\mathcal{W}, \otimes_{\mathcal{V}}) \rightarrow \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$$

is surjective.

Proof. Let $L \in \mathbb{C}^2$ be a generic line containing $\otimes_{\mathcal{V}}$. By the Lefschetz Hyperplane Theorem (see Corollary A.1), $\pi_1(\mathcal{W}, \otimes_{\mathcal{V}})$ is generated by the image of $\pi_1(L \cap \mathcal{W}, \otimes_{\mathcal{V}})$ under the map induced by inclusion

$$L \cap \mathcal{W} \hookrightarrow \mathcal{W},$$

and $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$ is generated by the image of $\pi_1(L \cap \widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$ under the map induced by inclusion

$$L \cap \widehat{\mathcal{W}} \hookrightarrow \widehat{\mathcal{W}}.$$

Let $j_L : L \cap \mathcal{W} \rightarrow L \cap \widehat{\mathcal{W}}$ be the restriction of j . The following diagram commutes.

$$\begin{array}{ccc} \pi_1(L \cap \mathcal{W}, \otimes_{\mathcal{V}}) & \longrightarrow & \pi_1(\mathcal{W}, \otimes_{\mathcal{V}}) \\ \downarrow (j_L)_* & & \downarrow j_* \\ \pi_1(L \cap \widehat{\mathcal{W}}, \otimes_{\mathcal{V}}) & \longrightarrow & \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}}). \end{array}$$

Consider the map

$$(j_L)_* : \pi_1(L \cap \mathcal{W}, \otimes_{\mathcal{V}}) \rightarrow \pi_1(L \cap \widehat{\mathcal{W}}, \otimes_{\mathcal{V}}).$$

There is a right inverse given by taking oriented loops around the points in $L \cap \widehat{\mathcal{W}}$ to oriented loops around the same points in $L \cap \mathcal{W}$, and the claim follows. \square

Abusing notation, denote by proj_x and proj_y the restrictions of the coordinate projections to $\widehat{\mathcal{W}}$.

Proposition 4.6. *We have*

$$\widehat{S} = \{\gamma \in \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}}) \mid (\text{proj}_x)_*(\gamma) = (\text{proj}_y)_*(\gamma)\}.$$

Proof. By definition, $\widehat{S} = j_*(S_f)$, and by Proposition 3.3

$$S_f = j_* \left(\{\gamma \in L_f \mid (\mu_A)_*(\gamma) = (\mu_A^B \circ \mu_B)_*(\gamma)\} \right).$$

By Proposition 4.1 we have the commutative diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{j} & \widehat{\mathcal{W}} \\ \downarrow \mu_B & & \downarrow \text{proj}_y \\ \mathcal{M}_B & \xrightarrow{\mu_A^B} & \mathcal{M}_A \end{array}$$

which implies the equality

$$(\text{proj}_y \circ j)_* = (\mu_A^B \circ \mu_B)_*,$$

and by Proposition 4.1, we have

$$(\text{proj}_x \circ j)_* = (\mu_A)_*.$$

The claim follows by surjectivity of j_* (see Proposition 4.5). \square

Remark 4.7. Our strategy is to produce an element $g \in \widehat{S}$ none of whose powers other than the identity is in \widehat{E} . It then follows that the group generated by any nontrivial element $\gamma \in j_*^{-1}(g) \subseteq S_f$ has infinite order, and

$$\langle \gamma \rangle \cap E_f = 1.$$

This implies that the cosets $\gamma^n \cdot E_f \subseteq S_f$ are distinct and hence the index of E_f in S_f is infinite. Theorem 1.3 then follows from Proposition 3.4.

4.4. Presentations. In this section we present the fundamental groups of $\widehat{\mathcal{W}}$ and $\widehat{\mathcal{V}}$ using standard braid monodromy techniques [MT], [H]. The space $\widehat{\mathcal{W}}$ is homeomorphic to the complement of the well-studied Ceva line arrangement and other computations of its fundamental group can be found, for example, in [ACL]. We repeat the computation here in order to make clear the relation with the fundamental group of $\widehat{\mathcal{V}}$.

In the following, all loops encircling punctures are oriented by the complex structure, and in the figures turn in the counter-clockwise direction.

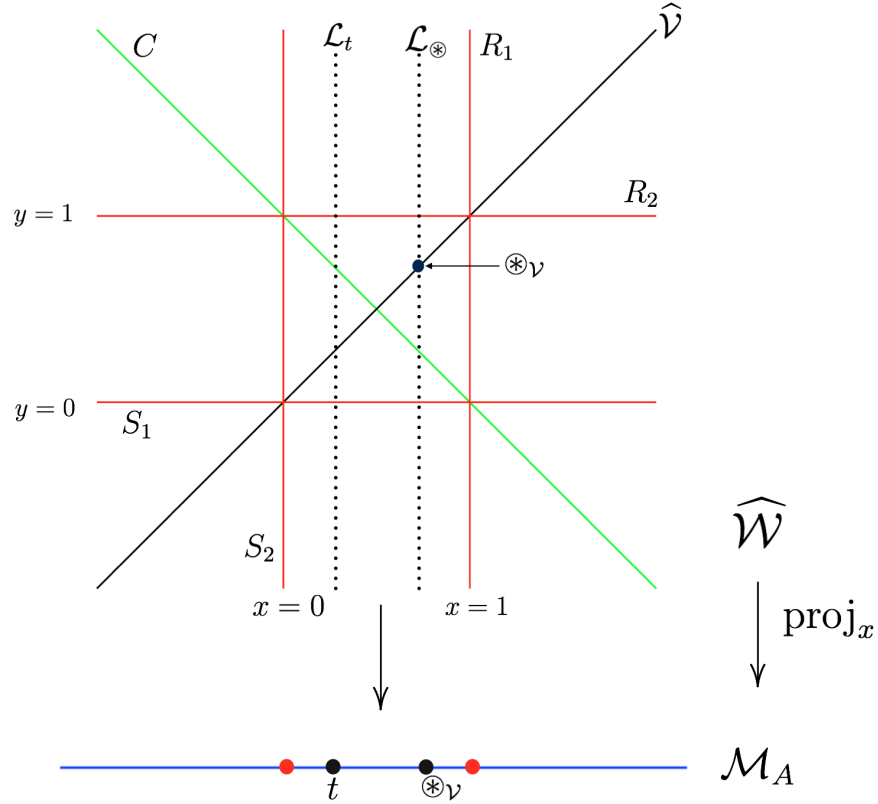


FIGURE 2. The space $\widehat{\mathcal{W}}$ is a fiber bundle over \mathcal{M}_A with each fiber equal to the complement in \mathbb{C} of 3 points given by the intersections of the lines S_1 , R_2 , and C .

The map $\text{proj}_x : \widehat{\mathcal{W}} \rightarrow \mathcal{M}_A$ is a fiber bundle, over a $K(\pi, 1)$. Here $\pi = \pi_1(\mathcal{M}_A, \otimes_A)$ is the free group on 2 generators. Thus, for $\mathcal{L}_\otimes := \text{proj}_x^{-1}(\otimes_A)$, we have a short exact sequence of fundamental groups

$$(4) \quad 1 \rightarrow \pi_1(\mathcal{L}_\otimes, \otimes_{\mathcal{V}}) \rightarrow \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}}) \rightarrow \pi_1(\mathcal{M}_A, \otimes_A) \rightarrow 1.$$

We can present $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$ as a semi-direct product

$$\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}}) = \pi_1(\mathcal{L}_\otimes, \otimes_{\mathcal{V}}) \rtimes F_2,$$

where F_2 is the image of a splitting $\pi_1(\mathcal{M}_A, \otimes_A) \hookrightarrow \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$.

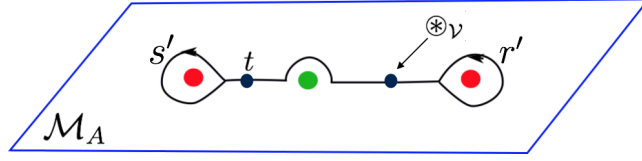


FIGURE 3. Representations of the generators r' and s' for $\pi_1(\mathcal{M}_A, \otimes_A)$ are drawn as in black, and encircle the images of the lines R_1 and S_2 . The central dot indicates the image of the line C .

Present the fundamental group $\pi_1(\mathcal{M}_A, \otimes_A)$ as the free group on the generating loops r' and s' drawn in Figure 3. The splitting will be defined as follows. Note that $\text{proj}_x|_{\widehat{\mathcal{V}}}$ injects $\widehat{\mathcal{V}}$ into \mathcal{M}_A . By definition, r' and s' lie in the image of $\text{proj}_x|_{\widehat{\mathcal{V}}}$ and hence have well-defined lifts r, s in $\widehat{\mathcal{V}}$. We take the map that sends the generators r' to r and s' to s to be our splitting $\pi_1(\mathcal{M}_A, \otimes_A) \rightarrow \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$. The fiber group $\pi_1(\mathcal{L}_{\otimes}, \otimes_{\mathcal{V}})$ is freely generated by r_2, s_1, c , the meridional loops on \mathcal{L}_{\otimes} around R_2, S_1 and C , respectively, as drawn in Figure 4.

Following the Zariski-van Kampen Algorithm (see Theorem A.1), we can write $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$ as a semi-direct product $\langle r_2, s_1, c \rangle \rtimes \langle r, s \rangle$, or

$$\langle r_2, s_1, c, r, s : r^{-1}r_2r = \phi_r(r_2), r^{-1}s_1r = \phi_r(s_1), s^{-1}r_2s = \phi_s(r_2), s^{-1}s_1s = \phi_s(s_1) \rangle$$

where the actions ϕ_r of r and ϕ_s of s by conjugation on $\langle r_2, s_1, c \rangle$ are defined by the monodromy of fibers over the loops r' and s' . To find the monodromy, we note that the local monodromy on a small loop encircling a singular value of the projection in the counter-clockwise direction can be represented by the braid that does a full counter-clockwise twist on the strands corresponding to the intersecting lines: the monodromy around a semi-circle in the counter-clockwise direction is a half counter-clockwise twist on the corresponding strands. Snapshots of particular fibers over the loops r' and s' are illustrated in Figure 4.

One finds ϕ_r and ϕ_s by comparing the snapshots of \mathcal{L}_{\otimes} in Figure 4 before and after the monodromies defined by r' and s' . These maps are indicated in Figure 4 by the arrows labeled ϕ_r and ϕ_s . The maps ϕ_r and ϕ_s give the following relations for $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$:

$$\begin{aligned} r^{-1}r_2r &= r_2 \\ r^{-1}cr &= r_2^{-1}s_1cs_1^{-1}r_2 \\ r^{-1}s_1r &= r_2^{-1}s_1cs_1c^{-1}s_1^{-1}r_2 \\ s^{-1}r_2s &= s_1^{-1}cr_2c^{-1}s_1 \\ s^{-1}cs &= s_1^{-1}cr_2cr_2^{-1}c^{-1}s_1 \\ s^{-1}s_1s &= s_1. \end{aligned}$$

Let $r_1 = rr_2^{-1}$ and $s_2 = ss_1^{-1}$. With respect to the new generators r_1, r_2, s_1, s_2, c , the presentation simplifies to

$$(5) \quad \langle r_1, r_2, s_1, s_2, c : r_1r_2 = r_2r_1, s_1s_2 = s_2s_1, r_1s_1c = cr_1s_1 = s_1cr_1, r_2s_2c = s_2cr_2 = cr_2s_2 \rangle.$$

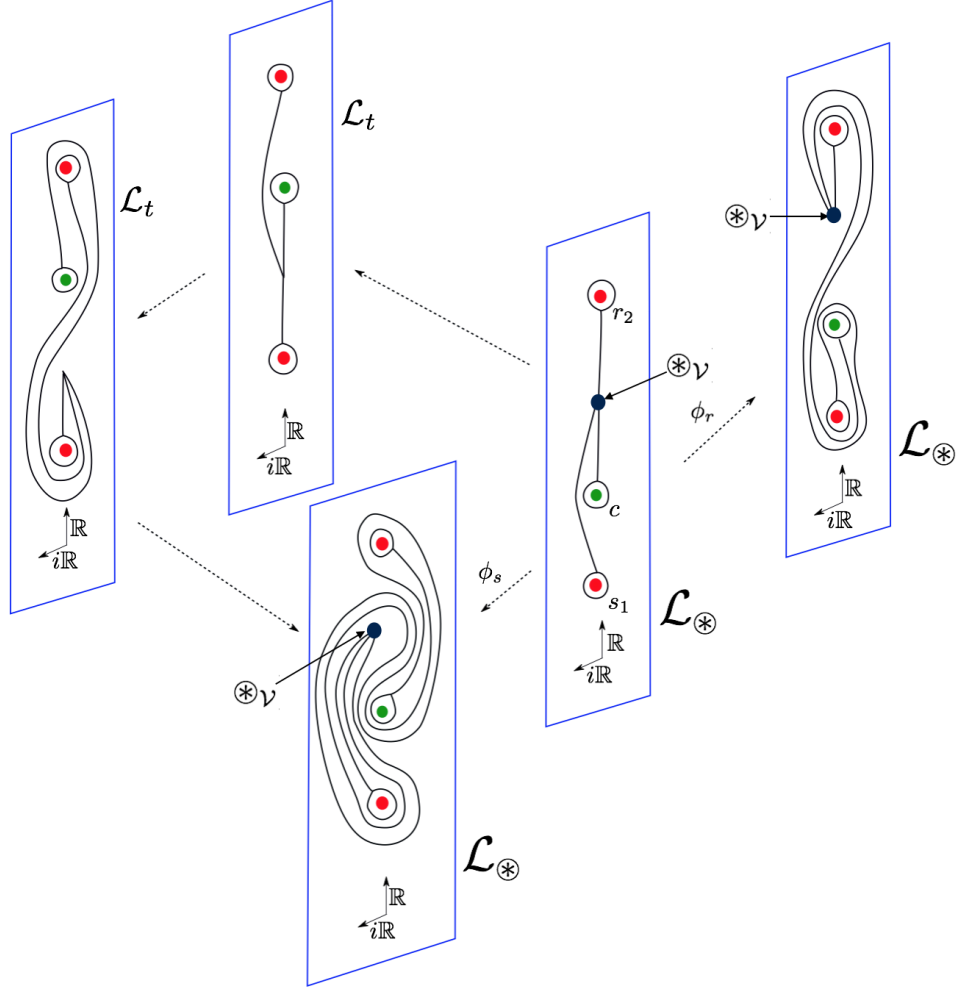


FIGURE 4. The monodromy actions ϕ_r and ϕ_s on generators of $\pi_1(\mathcal{L}_*, \otimes_\nu)$ are drawn. The action ϕ_r is split up into stages corresponding to the decomposition of r' into the path segments \otimes_A to t , a closed loop based at t and a return path segment t to \otimes_A (see Figure 2).

The group $\pi_1(\widehat{\mathcal{V}}, \otimes_\nu)$ is freely generated by r, s and c and their images under \widehat{i}_* are given by

$$\begin{aligned} r &\mapsto r = r_1 r_2 \\ s &\mapsto s = s_1 s_2 \\ c &\mapsto c. \end{aligned}$$

Proof of Theorem 1.3. Let $g = s_2 c r_1 s_1 c r_2$. We claim that $g \in \widehat{S}$, and no nonzero power of g lies in \widehat{E} . We have

$$(\text{proj}_x)_*(g) = s' r', \quad (\text{proj}_y)_*(g) = (\text{proj}_x)_*(s_1 c r_2 s_2 c r_1) = s' r'.$$

Consider the quotient Q of $\pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}})$ given by adding the relations $a = r_1 = r_2, b = s_1 = s_2$. Then we have

$$Q = \langle a, b, c, d : d = abc = bca = cab \rangle \simeq \langle d \rangle \times \langle a, b \rangle,$$

and quotient map $q : \pi_1(\widehat{\mathcal{W}}, \otimes_{\mathcal{V}}) \rightarrow Q$. If we add the relation $d = 0$, then

$$q(\widehat{E}) = \langle a^2, b^2, ab \rangle \subseteq \langle a, b \rangle,$$

is a free subgroup, and has trivial center.

Since $q(g) = d^2$ has infinite order in Q , g must have infinite order in \widehat{S} . On the other hand, the nonzero powers of $q(g)$ lie in the center of Q , while the center of \widehat{E} in Q is trivial. This implies that no nonzero power of g lies in \widehat{E} .

We have shown that g satisfies the conditions in Remark 4.7 thus proving Theorem 1.3. \square

APPENDIX A. ZARISKI-VAN KAMPEN ALGORITHM

A.1. Fundamental groups of complements of algebraic plane curves. In Sections 4.3 and 4.4, we will use a technique originally due to van Kampen and Zariski (see [vK], [C]) for computing presentations of the fundamental group of the complement of an algebraic plane curve $\mathcal{C} \subseteq \mathbb{C}^2$. For the reader's convenience, we give a brief outline of the technique for the special case of line arrangements defined over \mathbb{R} (cf. [H]).

Let $L = L_1 \cup \dots \cup L_k$ and $J = J_1 \cup \dots \cup J_s \subseteq \mathbb{C}^2$ be unions of distinct lines, and let $\text{proj} : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a projection such that

- the projection is generic with respect to L_1, \dots, L_k , in particular, no component L_i is equal to a fiber of proj , and
- each of the components J_1, \dots, J_s of J are fibers of proj .

The Zariski-van Kampen algorithm, which we will now describe, gives a way to compute $\pi_1(\mathbb{C}^2 - L \cup J, \bullet)$.

Let $U = \{u_1, \dots, u_r\} \subseteq \mathbb{C}$ be the images of the intersection points of L , and let $V = \{v_1, \dots, v_s\} \subseteq \mathbb{C}$ be the images of J_1, \dots, J_s under proj . Let $P = U \cup V$, $E = \mathbb{C}^2 - L \cup J$ and let $S = \text{proj}^{-1}(P)$. Then proj restricts to a fiber bundle

$$p : E - S \rightarrow \mathbb{C}.$$

Let $*$ be an arbitrary point in $\mathbb{C} - P$, and let $F_* = p^{-1}(*)$. Then F_* is isomorphic to a complex line in \mathbb{C}^2 with k points removed. Let \bullet be a point in F_* . Then, since $\mathbb{C} - P$ is a $K(\pi, 1)$, we have a short exact sequence

$$(6) \quad 1 \rightarrow \pi_1(F_*, \bullet) \rightarrow \pi_1(E - S, \bullet) \rightarrow \pi_1(\mathbb{C} - P, *) \rightarrow 1.$$

Write $\pi_1(F_*, \bullet) = \langle x_1, \dots, x_k \rangle$ and $\pi_1(\mathbb{C} - P, *) = \langle y'_1, \dots, y'_r, z'_1, \dots, z'_s \rangle$, where y'_1, \dots, y'_r are meridinal loops around u_1, \dots, u_r and z'_1, \dots, z'_s are meridinal loops around v_1, \dots, v_s . Here a meridinal loop in $\mathbb{C} - P$ around a point $w \in P$ is a loop that follows a path τ in $\mathbb{C} - P$ from $*$ to a point near w then follows a small circle in the counter-clockwise direction around w , then follows τ back to $*$. With this notation, $\pi_1(E - S, \bullet)$ is a semi-direct product

$$\pi_1(E - S, \bullet) = \pi_1(F_*, \bullet) \rtimes \alpha_*(\pi_1(\mathbb{C} - P, *))$$

where

$$\alpha : \pi_1(\mathbb{C} - P, *) \rightarrow \pi_1(E - S, \bullet)$$

$$y'_i \mapsto y_i$$

is a choice of splitting. Thus, $\pi_1(E - S, \bullet)$ has a presentation with generators

$$x_1, \dots, x_k, y_1, \dots, y_r, z_1, \dots, z_s$$

and relations

$$y_i^{-1} x_j y_i = \phi_i(x_j), i = 1, \dots, r, j = 1, \dots, k$$

and

$$z_i^{-1} x_j z_i = \psi_i(x_j), i = 1, \dots, r, j = 1, \dots, k.$$

where

$$\phi_1, \dots, \phi_r, \psi_1, \dots, \psi_s \in \text{Aut}(\pi_1(F_*, \bullet))$$

are determined by the choice of splitting α . An iteration of applications of the van Kampen theorem for fundamental groups of unions can be used to prove the following [vK] [C].

Theorem A.1 (Zariski-van Kampen Theorem for complements of planar line arrangements). *If J is a finite union of fibers of a generic projection of $\mathbb{C}^2 - L$ to \mathbb{C} , then the presentation of $\pi_1(\mathbb{C}^2 - (L \cup J), \bullet)$ is obtained from that of $\pi_1(E - S, \bullet)$ by adding the relations*

$$y_1 = 1, \dots, y_k = 1.$$

In the case when J is empty, we have the following consequence.

Corollary A.1 (Lefschetz Hyperplane Theorem for complements of planar line arrangements). *Given an arrangement of lines $L \subseteq \mathbb{C}^2$, and inclusion $F_* \hookrightarrow \mathbb{C}^2 - L$, where F_* is any line in general position with respect to L there is a surjective map on fundamental groups*

$$\pi_1(F_*, \bullet) \rightarrow \pi_1(\mathbb{C}^2 - L, \bullet).$$

where $\bullet \in F_* - F_* \cap L$ is any arbitrary element.

In practice, it is not difficult to find ϕ_i and ψ_j using local monodromies, particularly in the case when the lines are defined by real equations. This is because we can choose generators for $\pi_1(\mathbb{C} - P, *)$ that are concatenations of real segments, and small semi-circles or circles centered at the points in P . This allows one to decompose the monodromy into simpler pieces as is done in Section 4.4.

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